

# Alperin's Fusion Theorem for localities

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In these notes we give an Alperin's Fusion Theorem for localities.

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The notion of locality was introduced by Chermak [Ch] to study  $p$ -local structures of finite groups and to prove the existence and uniqueness of linking systems associated to saturated fusion systems. In the way he proved that there is a one-to-one correspondance between localities with saturated fusion system  $\mathcal{F}$  and transporter systems associated to  $\mathcal{F}$ . Linking systems and transporter systems were introduced by Broto, Levi and Oliver in [BLO2] and Oliver and Ventura in [OV1] respectively. They introduced these categories to study saturated fusion systems,  $p$ -completed classifying spaces of finite groups and to develop a theory of classifying spaces for saturated fusion systems. Localities give a more group-like point of view on these objects which allows us to use tools from group theory. This paper gives also an example where the setup of localities helps to prove results on transporter systems.

We present here a version of Alperin's Fusion Theorem for localities which generalizes the Alperin Fusion Theorem for finite groups (a nice version can be found in [St] Theorem 1). Chermak already gave an Alperin's Fusion Theorem for proper localities (i.e. localities which corresponds to linking systems) in [Ch] Proposition 2.17. Here to be able to work with any locality we have to relax a bit his notion of  $\mathcal{L}$ -essential subgroups. However, our definition of  $\mathcal{L}$ -essential corresponds to the original notion of "essential" (See [St] Definition page 97). We use this Theorem to get a generalization of the Alperin Fusion Theorem for transporter systems given by Oliver and Ventura in [OV1]. This may be applied for example with group cohomology.

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## 1 Localities and fusion systems

### 1.1 Partial groups, objective partial groups and localities

The notions of partial group and locality are due to Chermak. We present here the definitions and the useful properties but more details can be found in [Ch].

For  $X$  a set  $\mathbb{W}(X)$  will denote the free monoid on  $X$  and for two words  $u, v \in \mathbb{W}(X)$ ,  $u \circ v$  will denote the concatenation of  $u$  and  $v$ . We also identify  $X$  as the subset of words of length 1 in  $\mathbb{W}(X)$ .

**Definition 1.1** Let  $\mathcal{M}$  be a set and  $\mathbb{D}(\mathcal{M}) \subseteq \mathbb{W}(\mathcal{M})$  be a subset such that,

- (i)  $\mathcal{M} \subseteq \mathbb{D}(\mathcal{M})$ ; and
- (ii)  $u \circ v \in \mathbb{D}(\mathcal{M}) \Rightarrow u, v \in \mathbb{D}(\mathcal{M})$  (in particular,  $\emptyset \in \mathbb{D}(\mathcal{M})$ ).

A mapping  $\Pi : \mathbb{D}(\mathcal{M}) \rightarrow \mathcal{M}$  is a *product* if

- (P1)  $\Pi$  restricts to the identity on  $\mathcal{M}$ ; and
- (P2) if  $u \circ v \circ w \in \mathbb{D}(\mathcal{M})$  then  $u \circ \Pi(v) \circ w \in \mathbb{D}(\mathcal{M})$  and

$$\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w).$$

The *unit* of  $\Pi$  is then defined as  $\Pi(\emptyset)$ . A *partial monoid* is a triple  $(\mathcal{M}, \mathbb{D}(\mathcal{M}), \Pi)$  where  $\Pi$  is a product defined on  $\mathbb{D}(\mathcal{M})$ .

An *inversion* on  $\mathcal{M}$  is an involutory bijection  $x \mapsto x^{-1}$  on  $\mathcal{M}$  together with the induced mapping  $u \mapsto u^{-1}$  on  $\mathbb{W}(\mathcal{M})$  defined by,

$$u = (x_1, x_2, \dots, x_n) \mapsto (x_n^{-1}, x_{n-1}^{-1}, \dots, x_1^{-1}).$$

A *partial group* is a tuple  $(\mathcal{M}, \mathbb{D}(\mathcal{M}), \Pi, (-)^{-1})$  where  $\Pi$  is a product on  $\mathbb{D}(\mathcal{M})$  and  $(-)^{-1}$  is an inversion on  $\mathcal{M}$  satisfying

- (P3) if  $u \in \mathbb{D}(\mathcal{M})$  then  $(u^{-1}, u) \in \mathbb{D}(\mathcal{M})$  and  $\Pi(u^{-1} \circ u) = 1$ .

To simplify the notation, we will use  $\mathcal{M}$  to refer to a partial group  $(\mathcal{M}, \mathbb{D}(\mathcal{M}), \Pi, (-)^{-1})$  when the rest of the data is understood. For  $g \in \mathcal{M}$  we also write

$$\mathbb{D}(g) = \{x \in \mathcal{M} \mid (g^{-1}, x, g) \in \mathbb{D}\}.$$

**Example** Let  $\mathcal{M}$  be a partial group. if  $\mathbb{D}(\mathcal{M}) = \mathbb{W}(\mathcal{M})$  then  $\mathcal{M}$  is a group via the binary operation  $(x, y) \in \mathcal{M}^2 \mapsto \Pi(x, y) \in \mathcal{M}$ .

**Definition 1.2** Let  $\mathcal{L}$  be a partial group and  $\Delta$  a collection of subgroups of  $\mathcal{L}$ . Define  $\mathbb{D}_\Delta$  to be the set of all  $w = (g_1, g_2, \dots, g_n) \in \mathbb{W}(\mathcal{L})$  such that

- (\*) there exists  $(X_0, X_1, \dots, X_n) \in \mathbb{W}(\Delta)$  with  $(X_{i-1})^{g_i} = X_i$  for all  $i(1 \leq i \leq n)$ .

Then,  $(\mathcal{L}, \Delta)$  is an *objective partial group* if the following two conditions holds.

- (O1)  $\mathbb{D}(\mathcal{L}) = \mathbb{D}_\Delta$ .
- (O2) Whenever  $X$  and  $Y$  are in  $\Delta$  and  $g \in \mathcal{L}$  such that  $X^g$  is a subgroup of  $Y$ , then every subgroup of  $Y$  containing  $X^g$  is in  $\Delta$ .

**Definition 1.3** Let  $p$  be a prime, let  $\mathcal{L}$  be a finite partial group. Let  $S$  be a  $p$ -subgroup of  $\mathcal{L}$ , and let  $\Delta$  be a collection of subgroups of  $S$  such that  $S \in \Delta$ . Then,  $(\mathcal{L}, \Delta, S)$  is a *locality* if:

- (L1)  $(\mathcal{L}, \Delta)$  is objective; and
- (L2)  $S$  is maximal in the poset (ordered by inclusion) of finite  $p$ -subgroups of  $\mathcal{L}$ .

If  $(\mathcal{L}, \Delta, S)$  is a locality and  $g \in \mathcal{L}$ , we write

$$S_g = \{s \in S \mid (g^{-1}, x, g) \in \mathbb{D}(\mathcal{L}) \text{ and } x^g \in S\}.$$

**Lemma 1.4** Let  $(\mathcal{L}, \Delta, S)$  be a locality.

- (a) For every  $X \in \Delta$ ,  $N_{\mathcal{L}}(X)$  is a subgroup of  $\mathcal{L}$ .
- (b) Let  $g \in \mathcal{L}$  and let  $X \in \Delta$  with  $X^g \in \Delta$ . Then  $N_{\mathcal{L}}(X) \subseteq \mathbb{D}(g)$ , and

$$c_g : N_{\mathcal{L}}(X) \longrightarrow N_{\mathcal{L}}(X^g)$$

is an isomorphism of groups.

- (c) every  $p$ -subgroup of  $\mathcal{L}$  is conjugate to a subgroup of  $S$ .

**Proof** Propositions (a) and (b) correspond to propositions (a) and (b) of [Ch] Lemma 2.7 and are in fact true in an objective partial group. Finally, (c) is proved in [Ch] Proposition 2.22 (b).  $\square$

## 1.2 Fusion systems

A fusion system over a  $p$ -group  $S$  is a way to abstract the action of a finite group  $G \geq S$  on the subgroups of  $S$  by conjugation. For  $G$  a finite group and  $g \in G$ , we will denote by  $c_g$  the homomorphism  $x \in G \mapsto g^{-1}xg \in G$  and for  $H, K$  two subgroups of  $G$ ,  $\text{Hom}_G(H, K)$  will denote the set of all group homomorphisms  $c_g$  for  $g \in G$  such that  $c_g(H) \leq K$  and  $\text{Syl}_p(G)$  will denote the collection of Sylow  $p$ -subgroup of  $G$ .

**Definition 1.5** Let  $S$  be a finite  $p$ -group. A *fusion system* over  $S$  is a small category  $\mathcal{F}$ , where  $\text{Ob}(\mathcal{F})$  is the set of all subgroups of  $S$  and which satisfies the following two properties for all  $P, Q \leq S$ :

- (a)  $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$ ;
- (b) each  $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$  is the composite of an  $\mathcal{F}$ -isomorphism followed by an inclusion.

The composition in a fusion system is given by composition of homomorphisms. We usually write  $\text{Hom}_{\mathcal{F}}(P, Q) = \text{Mor}_{\mathcal{F}}(P, Q)$  to emphasize that the morphisms in  $\mathcal{F}$  are homomorphisms.

The typical example of fusion system is the fusion system of a finite group  $G$ .

**Example** Let  $G$  be a finite group and  $S \in \text{Syl}_p(G)$ . The *fusion system of  $G$  over  $S$*  is the category  $\mathcal{F}_S(G)$  where  $\text{Ob}(\mathcal{F}_S(G))$  is the set of all subgroups of  $S$  and for all  $P, Q \leq S$ ,  $\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q)$ . The category  $\mathcal{F}_S(G)$  defines a fusion system over  $S$ .

An other important example is the fusion system of a locality.

**Example** Let  $(\mathcal{L}, \Delta, S)$  be a locality. The *fusion system of  $\mathcal{L}$  over  $S$*  is the fusion system  $\mathcal{F}_S(\mathcal{L})$  generated by  $\{c_g : S_g \rightarrow (S_g)^g \mid g \in \mathcal{L}\}$ : for  $P, Q \leq S$  and  $\varphi \in \text{Hom}(P, Q)$ ,  $\varphi \in \text{Hom}_{\mathcal{F}_S(\mathcal{L})}(P, Q)$  if  $\varphi$  is a composite of restrictions of  $c_g : S_g \rightarrow (S_g)^g$  for  $g \in \mathcal{L}$ .

**Definition 1.6** Let  $S$  be a  $p$ -group and  $\mathcal{F}$  a fusion system over  $S$ .

- (a) Two subgroups  $P, Q \leq S$  are  $\mathcal{F}$ -conjugate if they are isomorphic as objects in  $\mathcal{F}$ . We denote by  $P^{\mathcal{F}}$  the set of all subgroups of  $S$   $\mathcal{F}$ -conjugate to  $P$ .

- (b) A subgroup  $P \leq S$  is *fully normalized* (resp. *fully centralized*) if for every  $Q \in P^{\mathcal{F}}$ ,  $|N_S(P)| \geq |N_S(Q)|$  (resp.  $|C_S(P)| \geq |C_S(Q)|$ ).
- (c) A subgroup  $P \leq S$  is *receptive* in  $\mathcal{F}$  if it has the following property: for each  $Q \leq S$  and  $\varphi \in \text{Iso}_{\mathcal{F}}(Q, P)$ , if we set

$$N_{\varphi} = \{g \in N_S(Q) \mid \varphi \circ c_g \circ \varphi^{-1} \in \text{Aut}_S(P)\},$$

then there is  $\bar{\varphi} \in \text{Hom}_{\mathcal{F}}(N_{\varphi}, S)$  such that  $\bar{\varphi}|_Q = \varphi$ .

**Definition 1.7** Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ . Let  $\Delta$  be a collection of subgroup of  $S$ .

- (a)  $\Delta$  is  $\mathcal{F}$ -invariant if for every  $P \in \Delta$  and  $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ ,  $\varphi(P) \in \Delta$ .
- (b)  $\Delta$  is  $\mathcal{F}$ -closed if it is  $\mathcal{F}$ -invariant and closed by passing to overgroups in  $S$ .

For example, if  $(\mathcal{L}, \Delta, S)$  is a locality,  $\Delta$  is  $\mathcal{F}_S(\mathcal{L})$ -invariant and  $\mathcal{F}_S(\mathcal{L})$ -closed.

**Definition 1.8** Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$  and let  $\Delta$  be an  $\mathcal{F}$ -invariant collection of subgroups of  $S$ . We say that  $\mathcal{F}$  is  $\Delta$ -saturated if

- (I) If  $P \in \Delta$  is fully normalized, then it is also fully centralized and  $\text{Aut}_S(P)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ ; and
- (II) If  $P \in \Delta$  is fully centralized, then it is also receptive.

We say that  $\mathcal{F}$  is *saturated* if  $\mathcal{F}$  is  $\Delta$ -saturated for  $\Delta$  the set of all subgroups of  $S$ .

Saturation of a fusion system  $\mathcal{F}$  over a  $p$ -group  $S$  can be controlled by a smaller collection of subgroups than all the subgroups of  $S$ .

**Definition 1.9** Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ .

- (a) A subgroup  $P \leq S$  is  $\mathcal{F}$ -centric if, for every  $Q \in P^{\mathcal{F}}$ ,  $C_S(Q) = Z(Q)$ .
- (b) A subgroup  $P \leq S$  is  $\mathcal{F}$ -radical if  $O_p(\text{Out}_{\mathcal{F}}(P)) = 1$ .

We denote by  $\mathcal{F}^c$ ,  $\mathcal{F}^r$  and  $\mathcal{F}^{cr}$  the collections of, respectively,  $\mathcal{F}$ -centric,  $\mathcal{F}$ -radical and,  $\mathcal{F}$ -centric and  $\mathcal{F}$ -radical subgroups of  $S$ .

**Theorem 1.10** ([5a1], Theorem 2.2) *Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ . Let  $\Delta$  be an  $\mathcal{F}$ -closed collection of subgroups of  $S$  such that  $\mathcal{F}^{cr} \subseteq \Delta$ . Suppose that  $\mathcal{F}$  is  $\Delta$ -saturated and that each  $\mathcal{F}$ -isomorphism is a composite of restrictions of  $\mathcal{F}$ -isomorphisms between members of  $\Delta$ . Then  $\mathcal{F}$  is saturated.*

**Proposition 1.11** ([Ch], Proposition 2.17) *Let  $(\mathcal{L}, \Delta, S)$  be a locality. Then,  $\mathcal{F}_S(\mathcal{L})$  is  $\Delta$ -saturated.*

In particular, if  $\mathcal{F}^{cr} \subseteq \Delta$ , then  $\mathcal{F}_S(\mathcal{L})$  is saturated.

## 2 Alperin's Fusion Theorem

**Definition 2.1** Let  $G$  be a finite group. Let  $\Gamma_p(G)$  be the graph whose vertices are the Sylow  $p$ -subgroups of  $G$ , and those edges are the pairs  $\{S_1, S_2\}$  of distinct Sylow  $p$ -subgroups of  $G$  such that  $S_1 \cap S_2 \neq 1$ . We say that  $G$  is  $p$ -disconnected if  $\Gamma_p(G)$  is disconnected.

Let  $G$  be a finite group. Then  $G$  is acting on  $\Gamma_p(G)$  by conjugation, and, by Sylow's Theorem, this action is transitive on vertices. If  $S$  is a fixed Sylow  $p$ -subgroup of  $G$ , let  $\Sigma$  be the connected component of  $\Gamma_p(G)$  which contains  $S$  and let  $H$  be the set-wise stabilizer of  $\Sigma$  in  $G$ . If  $G$  is  $p$ -disconnected then  $H < G$  and for every  $g \in G \setminus H$ ,  $p$  does not divide  $|H \cap H^g|$ . Recall that a subgroup  $H$  of  $G$  is *strongly  $p$ -embedded* if  $H < G$  and for every  $g \in G \setminus H$ ,  $p$  does not divide  $|H \cap H^g|$ . Hence, if  $G$  is  $p$ -disconnected then  $G$  contains a strongly  $p$ -embedded subgroup. The converse is also true and we have the following remark.

**Remark 2.2** A finite group  $G$  is  $p$ -disconnected if and only if  $G$  contains a strongly  $p$ -embedded subgroup.

**Remark 2.3** Let  $G$  be a finite group and let  $\mathcal{S}_p^*(G)$  be the poset of nontrivial  $p$ -subgroup of  $G$ . Notice that  $\Gamma_p(G)$  is connected if and only if  $\mathcal{S}_p^*(G)$  is connected. Thus we could have defined  $p$ -disconnected with  $\mathcal{S}_p^*(G)$  instead of  $\Gamma_p(G)$ . But working with  $\Gamma_p(G)$  will be more convenient in the proof of Theorem 2.5.

**Definition 2.4** Let  $(\mathcal{L}, \Delta, S)$  be a locality.

- (a) A subgroup  $P \in \Delta$  is  $\mathcal{L}$ -radical if  $O_p(N_{\mathcal{L}}(P)/P) = 1$ .
- (b) A subgroup  $P \in \Delta$  is  $\mathcal{L}$ -essential if  $N_S(P) \in \text{Syl}_p(N_{\mathcal{L}}(P))$  and  $N_{\mathcal{L}}(P)/P$  is  $p$ -disconnected.

Denote  $\Delta^r$  and  $\Delta^e$  the subcollections of  $\Delta$  of, respectively,  $\mathcal{L}$ -radical and  $\mathcal{L}$ -essential subgroups of  $S$ .

Notice that, by Lemma 1.4, if  $P \leq S$  then  $N_S(P) \in \text{Syl}_p(N_{\mathcal{L}}(P))$  if and only if  $P$  is fully normalized in  $\mathcal{F}_S(\mathcal{L})$ .

Notice also that for a subgroup  $P \leq S$  to be  $\mathcal{L}$ -essential, we do not require  $P$  to be centric in  $\mathcal{L}$  as required in [Ch] Definition 2.4.

If  $G$  is a finite group,  $O_p(G) \neq 1$  implies that  $\Gamma_p(G)$  is connected. Therefore, if  $(\mathcal{L}, \Delta, S)$  is a locality, If  $P$  is  $\mathcal{L}$ -essential then  $P$  is  $\mathcal{L}$ -radical. But the converse is clearly false.

**Theorem 2.5** Let  $(\mathcal{L}, \Delta, S)$  be a locality. Then, for every  $g \in \mathcal{L}$ , there exists  $Q_1, Q_2, \dots, Q_n \in \Delta^e \cup \{S\}$  and  $w = (g_1, g_2, \dots, g_n) \in \mathbb{D}(\mathcal{L})$  such that,

- (i) for every  $i \in \{1, 2, \dots, n\}$ ,  $g_i \in N_{\mathcal{L}}(Q_i)$  and  $S_{g_i} = Q_i$ ; and
- (ii)  $S_w = S_g$  and  $g = \Pi(w)$ .

**Proof** We will say that  $g \in \mathcal{L}$  admits an *essential decomposition* if there exists  $Q_1, Q_2, \dots, Q_n \in \Delta^e \cup \{S\}$  and  $w = (g_1, g_2, \dots, g_n) \in \mathbb{D}(\mathcal{L})$  such that (i) and (ii) are satisfied. Notice that we have the followings.

- (1) If  $g \in \mathcal{L}$  admits an essential decomposition then  $g^{-1}$  admits an essential decomposition.
- (2) If  $(g, h) \in \mathbb{D}(\mathcal{L})$  with  $S_{(g,h)} = S_{gh}$  and,  $g$  and  $h$  admit an essential decomposition, then,  $gh$  admits an essential decomposition.

Assume Theorem 2.5 is false and, among all  $g \in \mathcal{L}$  which does not admit an essential decomposition, choose  $g$  with  $S_g$  as large as possible. Set  $P = S_g$ ,  $P' = P^g$  and  $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ .

If  $P = S$ , then  $g \in N_{\mathcal{L}}(S)$  and  $g$  admits an essential decomposition with  $n = 1$ ,  $Q_1 = S$  and  $w = (g)$ . Thus, we can assume that  $P < S$ . By Proposition 1.11,  $\mathcal{F}$  is  $\Delta$ -saturated. then, there is  $Q \in \Delta$  fully normalized and  $h, h' \in \mathcal{L}$  such that  $P^h = Q$  and  $(P')^{h'} = Q$ . By Lemma 1.4 and Sylow's Theorem (applied in  $N_{\mathcal{L}}(Q)$ ), we can choose  $h$  and  $h'$  such that  $N_S(P)^h \leq N_S(Q)$  and  $N_S(P')^{h'} \leq N_S(Q)$ . Then, by maximality of  $P$ ,  $h$  and  $h'$  admit an essential decomposition. The word  $w = (h^{-1}, g, h')$  is in  $\mathbb{D}(\mathcal{L})$  via  $Q$  and  $g' := h^{-1}gh' \in N_{\mathcal{L}}(Q)$ . Then  $g = hg'h'^{-1}$  and  $S(h, g', h'^{-1}) = S_g$ . Therefore, if  $g'$  admits an essential decomposition, then, by (1) and (2),  $g$  admits an essential decomposition, and we can assume that  $P$  is fully normalized and  $g \in N_{\mathcal{L}}(P)$ .

If  $P \in \Delta^e$ , then  $g$  admits an essential decomposition with  $n = 1$ ,  $Q_1 = P$  and  $w = (g)$ . Thus,  $P \in \Delta \setminus \Delta^e$  and  $\Gamma_p(N_{\mathcal{L}}(P))/P$  is  $p$ -connected. In particular, there is a sequence  $X_0, X_1, \dots, X_r$  of Sylow  $p$ -subgroup of  $N_{\mathcal{L}}(P)$  with  $X_0 = N_S(P)$ ,  $X_r = N_S(P)^g$  and with  $X_i$  adjacent in  $\Gamma_p(N_{\mathcal{L}}(P))$  to  $X_{i-1}$  for all  $i \in \{1, 2, \dots, r\}$ . Choose  $u \in N_{\mathcal{L}}(P)$  such that  $(X_0)^u = X_{r-1}$  and let  $v = u^{-1}g$ . We have  $(u, v) \in \mathbb{D}(\mathcal{L})$ ,  $g = uv$  and  $P \leq S_{(u,v)} \leq S_g = P$ . Thus,  $S_{(u,v)} = P$  and if  $u$  and  $v$  admit an essential decomposition, by (2),  $g$  admits an essential decomposition. Therefore, we can assume that  $(N_S(P) \cap N_S(P)^g)/P \neq 1$ . In particular,  $P < N_S(P) \cap N_S(P)^g \leq S_g = P$  which gives a contradiction.  $\square$

As mention before, for a subgroup  $P \leq S$  to be  $\mathcal{L}$ -essential, we do not require  $P$  to be centric. indeed, Theorem 2.5 does not work if we add this requirement.

**Example** Let  $\mathcal{L} = \Sigma_3$  be the symmetric group over 3 letter and  $p = 2$ . let  $S = \langle (1, 2) \rangle$  and  $\Delta$  be the collection of all the subgroups of  $S$  (i.e.  $\Delta = \{S, \{e\}\}$ ). Then  $(\mathcal{L}, \Delta, S)$  is a locality where  $S$  and  $\{e\}$  are  $\mathcal{L}$ -essential. But only  $S$  is centric and  $(1, 2, 3)$  is not the product of elements in  $N_{\mathcal{L}}(S) = S$ .

### 3 Application to transporter systems

#### 3.1 Transporter systems and localities

We refer the reader to [OV1] Definition 3.1 for the definition of transporter system. The typical example is the following.

**Example** Let  $G$  be a finite group  $S \in \text{Syl}_p(G)$  and  $\Delta$  an  $\mathcal{F}$ -invariant collection of subgroups of  $S$ . The *transporter category* of  $G$  over  $S$  with set of object  $\Delta$  is the small category  $\mathcal{T} = \mathcal{T}_S^{\Delta}(G)$  with set of object  $\Delta$  and, for  $P, Q \in \Delta$ ,

$$\text{Mor}_{\mathcal{T}}(P, Q) = \{g \in G \mid P^g \leq Q\}.$$

By [OV1] Proposition 3.12, it is a transporter system associated to  $\mathcal{F}_S(G)$ .

In [Ch] Appendix X, Chermak gives a one-to-one correspondence between localities with fusion system  $\mathcal{F}$  and transporter systems associated to  $\mathcal{F}$ .

**Definition 3.1** Let  $(\mathcal{L}, \Delta)$  be an objective partial group. We define the *transporter system* of  $(\mathcal{L}, \Delta)$  as the category  $\mathcal{T}_\Delta(\mathcal{L})$  with set of object  $\Delta$  and with, for  $P, Q \in \Delta$ ,

$$\text{Mor}_{\mathcal{T}}(P, Q) = \{g \in \mathcal{L} \mid P \leq S_g \text{ and } P^g \leq Q\}.$$

**Proposition 3.2** ([Ch], Lemma X.1) Let  $(\mathcal{L}, \Delta, S)$  be a locality with  $\mathcal{F}_S(\mathcal{L})$  saturated. Then  $\mathcal{T}_\Delta(\mathcal{L})$  defines a transporter system associated to  $\mathcal{F}_S(\mathcal{L})$ .

**Proposition 3.3** ([Ch], Proposition X.9) Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ , and  $\mathcal{T}$  be an associated transporter system. Then there exists a locality  $(\mathcal{L}, \Delta, S)$  such that  $\mathcal{T} = \mathcal{T}_\Delta(\mathcal{L})$ .

### 3.2 Alperin's Fusion system for transporter systems

In the following Definition, we recall the definition of  $\mathcal{T}$ -radical subgroup from [OV1], and we give the definition of  $\mathcal{T}$ -essential subgroup.

**Definition 3.4** Let  $\mathcal{F}$  be a saturated fusion system and  $\mathcal{T}$  be an associated transporter system.

- (i) A subgroup  $P \in \text{Ob}(\mathcal{T})$  is  $\mathcal{T}$ -radical if  $O_p(\text{Aut}_{\mathcal{T}}(P)/\delta_P(P)) = 1$ .
- (ii) A subgroup  $P \in \text{Ob}(\mathcal{T})$  is  $\mathcal{T}$ -essential if  $N_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{T}}(G))$  and  $\text{Aut}_{\mathcal{T}}(P)/P$  is  $p$ -disconnected.

Denote  $\mathcal{T}^r$  the full subcategories of  $\mathcal{T}$  with set of objects the  $\mathcal{T}$ -radical subgroups of  $S$  and  $\mathcal{T}^e$  the full subcategory of  $\mathcal{T}$  with set of objects  $S$  and all the  $\mathcal{T}$ -essential subgroups of  $S$ .

Let  $(\mathcal{L}, \Delta, S)$  be a locality. then a subgroup  $P \leq S$  is  $\mathcal{T}_\Delta(\mathcal{L})$ -radical (resp.  $\mathcal{T}_\Delta(\mathcal{L})$ -essential) if and only if  $P$  is  $\mathcal{L}$ -radical (resp.  $\mathcal{L}$ -essential).

The following Theorem gives a generalization of the Alperin Fusion Theorem for transporter systems ([OV1] Proposition 3.10).

**Theorem 3.5** Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ . If  $\mathcal{T}$  is a transporter system associated to  $\mathcal{F}$ , then every morphism in  $\mathcal{T}$  is a composite of restrictions of automorphisms of  $S$  or  $\mathcal{T}$ -essential subgroups.

**Proof** By Proposition 3.3, there is a locality  $(\mathcal{L}, \Delta, S)$  such that  $\mathcal{T} = \mathcal{T}_\Delta(\mathcal{L})$ . Let  $P, Q \in \Delta = \text{Ob}(\mathcal{T})$  and choose  $g \in \text{Mor}_{\mathcal{T}}(P, Q) \subseteq \mathcal{L}$ . Without lost of generality, we can assume that  $P = S_g$  and  $Q = S_g^g$ . By Theorem 2.5, there exists  $Q_1, Q_2, \dots, Q_n \in \Delta^e \cup \{S\}$  and  $w = (g_1, g_2, \dots, g_n) \in \mathbb{D}(\mathcal{L})$  such that,

- (i) for every  $i \in \{1, 2, \dots, n\}$ ,  $g_i \in N_{\mathcal{L}}(Q_i)$  and  $S_{g_i} = Q_i$ ; and
- (ii)  $S_w = S_g$  and  $g = \Pi(w)$ .

We have  $P = S_g = S_w \leq S_{g_1} = Q_1$  and inductively, for  $1 \leq i \leq n-1$ ,  $P^{g_1 \cdots g_i} \leq Q_{i+1} = S_{g_{i+1}}$ . Thus  $g$  is the composite of the restriction of  $g_i \in \text{Aut}_{\mathcal{T}}(Q_i)$  to  $g_i \in \text{Mor}_{\mathcal{T}}(P^{g_1 \cdots g_{i-1}}, P^{g_1 \cdots g_i})$ , for  $1 \leq i \leq n$ .  $\square$

The following Corollary, which is a direct consequence of Theorem 3.5, may be helpful when computing limits over transporter systems.



**Corollary 3.6** *Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ . Let  $\mathcal{T}$  be a transporter system associated to  $\mathcal{F}$  and let*

$$F : \mathcal{T} \longrightarrow \mathcal{A}$$

*be a functor into an abelian category  $\mathcal{A}$ . Then,*

$$\varprojlim_{\mathcal{T}} F = \varprojlim_{\mathcal{T}^e} F.$$

We can for example use the previous corollary with  $\mathcal{F} = \mathcal{F}_S(G)$ , where  $G$  is a finite group and  $S \in \text{Syl}_p(G)$ , and  $F = H^*(-, M)$  for  $M$  a  $\mathbb{Z}_{(p)}[G]$ -module.

**Corollary 3.7** *Let  $G$  be a finite group and  $S \in \text{Syl}_p(G)$ . Let  $M$  a  $\mathbb{Z}_{(p)}[G]$ -module. Then*

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S^e(G)} H^*(-, M).$$

**Proof** By the Cartan-Eilenberg Theorem ([CE] Chap XII, Theorem 10.1),

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S(G)} H^*(-, M).$$

Then we can apply Corollary 3.6. □

A more general version was actually proved by Grodal in [Gr] Corollary 10.4 using deep algebraic topology. The proof we give here is algebraic, more elementary and use directly the  $p$ -local structure of  $G$ .

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